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Heavy Meson Electromagnetic Mass Differences from QCD

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Abstract

We compute the electromagnetic mass differences of mesons containing a single heavy quark in terms of measurable data using QCD-based arguments in heavy-quark effective theory. We derive an unsubtracted dispersion relation that shows that the mass differences are calculable in terms of the properties of the lowest-lying physical intermediate states. We then consider the problem in the large-N limit, where N is the number of QCD colors. In this limit, we can write a kind of double-dispersion relation for the amplitude required to determine the electromagnetic mass difference. We use this to derive analogs of the Weinberg sum rules for heavy meson matrix elements valid to leading order in $1/N$ and to $O(1/m_Q)$ in the heavy quark expansion. In order to obtain our final result, we assume that the electromagnetic mass differences and sum rules are dominated by the lowest-lying states in analogy with the situation for the π^+–π^0 mass difference. Despite the fact that some of the matrix elements appearing in our final result have not yet been accurately measured, we can obtain useful estimates: for example, we obtain $(M_{B^+} - M_{B^0})^{\mathrm{EM}} \simeq +1.8$ MeV. We argue that our results are accurate to about 30%.

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1. Introduction

The computation of the electromagnetic mass differences of hadrons is one of the classic problems in strong interaction physics (see *e.g.* [1][2][3]; for reviews, see [4][5][6]). At one time, it was believed that electromagnetism was the only source of isospin breaking, so that computations of electromagnetic mass differences could be compared directly with experiment. With the advent of QCD, it is now understood that isospin breaking arises both from electromagnetism and from differences in the u and d quark current masses. The modern motivation for computing electromagnetic mass differences is to disentangle the electromagnetic contributions to isospin-violating mass differences from those of the current quark masses in order to obtain information about the current quark masses.*

In this paper, we will compute the $O(e^2)$ electromagnetic mass differences of lowest-lying mesons with quantum numbers $Q\bar{u}$ and $Q\bar{d}$ (denoted here by $P_{\bar{\ell}}$ ($\ell = u$ or d). We work to $O(1/m_Q)$ in the heavy-quark expansion and to leading order in the $1/N$ expansion, where N is the number of QCD colors. We give a detailed exposition of the formalism used and give useful estimates of the electromagnetic mass differences.† A detailed comparison to experiment and the extraction of information about the light quark masses is carried out in a separate paper [8].

We follow as closely as possible the method of the classic calculation of the π^+–π^0 mass difference [2][6]. The basic strategy is to write the $O(e^2)$ self-energy of the meson in terms of the forward Compton amplitude T, and then write a dispersion relation for T expressing it in terms of measurable data. In the case of the π^+–π^0 mass difference, one can use the fact that the pions are pseudo-Nambu–Goldstone bosons to write T in terms of *vacuum* current correlation functions depending only on a single kinematic invariant. Dispersion relations then relate these to data measured in $e^+e^- \to$ hadrons and τ decays. For heavy mesons we must work directly with T, which depends on two kinematic invariants. This makes the computation of the heavy meson electromagnetic mass differences more complicated than the π^+–π^0 mass difference.

We begin in section 2 by writing an unsubtracted dispersion relation that shows that T is determined by the properties of low-lying meson states and the lowest-lying excitations of the heavy–light quark system. Unfortunately, this dispersion relation cannot be used directly to compute T; one reason is that it depends on structure functions for timelike photon momenta, which are not measurable in practice. In section 3, we consider the problem in the combined heavy-quark and large-N limits. In the large-N limit, we can write T in terms of heavy-meson form factors and meson scattering amplitudes. We derive sum rules relating the properties of the masses and matrix elements of the states that determine T. (These are exact analogs of the Weinberg sum rules for vector- and axial-vector correlation functions in the vacuum [9].) By assuming that these sum rules are approximately saturated by the lowest-lying states, we can compute the heavy meson electromagnetic mass differences in terms of heavy meson form factors. This is analogous to the successful classic calculation of the π^+–π^0 mass difference. In section 4, we consider the $1/m_Q$ corrections to these results, and section 5 summarizes our results and gives our conclusions. This is a rather long paper, but the reader can get the main ideas by reading sections 2 and section 3 through subsection 3.1, followed by section 5

* The π^+–π^0 mass difference is a special case, since quark masses do not contribute to it at leading order. Therefore, the electromagnetic contribution is expected to dominate, and it can be compared directly with experiment.

† The electromagnetic mass differences of heavy mesons have also been considered in ref. [7], but the methods used are not based on a systematic approximation of QCD.

(which contains a summary of the main results). The sections omitted in this way consist mainly of repeated application of the ideas in the first part of the paper.

Our final result is similar to the prescription used long ago to compute baryon electromagnetic mass differences [1], but we emphasize that this prescription was never put on a firm foundation [4][5]. In fact, it is ironic that we are not able to extend our results to baryons because the large-N limit is more complicated for baryons.

2. Technical Preliminaries

In this section, we derive some technical results that provide the foundation for the rest of the paper. Although we focus on the electromagnetic mass differences of heavy mesons in this paper, most of the formalism in this section can be applied to any type of hadron.

2.1. Renormalization and Finiteness

The $O(e^2)$ electromagnetic contribution to the mass difference of the pseudoscalar mesons $P_{\bar{\ell}}$ ($\ell = u, d$) with flavor quantum numbers $Q\bar{\ell}$ is given by

$$\begin{aligned} \Delta M \equiv M_{\bar{u}} - M_{\bar{d}} = \frac{ie^2}{2} \int \frac{d^4 q}{(2\pi)^4} \frac{\Delta T(p,q)}{q^2 + i0+} \\ - \langle P_{\bar{u}}(p) | \delta\mathcal{L} | P_{\bar{u}}(p) \rangle + \langle P_{\bar{d}}(p) | \delta\mathcal{L} | P_{\bar{d}}(p) \rangle, \end{aligned} \tag{2.1}$$

where $\delta\mathcal{L}$ is the $O(e^2)$ counterterm in the underlying lagrangian required to render the result finite (see below); M is the mass of the heavy mesons in the absence of electromagnetic interactions; we have defined

$$\Delta T \equiv T_{\bar{u}} - T_{\bar{d}}, \tag{2.2}$$

where

$$T_{\bar{\ell}}(p,q) \equiv i \int d^4 x\, e^{iq\cdot x} \langle P_{\bar{\ell}}(p) | T J^\mu(0) J_\mu(x) | P_{\bar{\ell}}(p) \rangle. \tag{2.3}$$

Here, J^μ is the electromagnetic current, so T is a trace over the forward Compton amplitude for scattering of photons from heavy mesons. For states containing heavy mesons, we use the normalization

$$\langle P(p) | P(p') \rangle = (2\pi)^3 \delta^3(\vec{p} - \vec{p}\,') \tag{2.4}$$

appropriate for the heavy particle effective theory. Similar equations hold for the lowest-lying heavy-light vector mesons P^* that are related to P by heavy-quark symmetry.

We now consider the counterterm $\delta\mathcal{L}$ in eq. (2.1). Because we will use the heavy-quark expansion, the underlying lagrangian is that of heavy quark effective theory [10]. To $O(1/m_Q)$ in the heavy-quark expansion, the lagrangian is

$$\begin{aligned} \mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - \frac{1}{4} \operatorname{tr}(G^{\mu\nu} G_{\mu\nu}) + \bar{q} i \not{D} q + \overline{Q} i v \cdot D Q \\ - \frac{1}{2m_Q} \overline{Q} D^2 Q + \frac{g_s a}{4 m_Q} \overline{Q} \sigma^{\mu\nu} G_{\mu\nu} Q + \frac{e_Q b}{4 m_Q} F^{\mu\nu} \overline{Q} \sigma_{\mu\nu} Q + \cdots. \end{aligned} \tag{2.5}$$

Here, $F^{\mu\nu}$ is the electromagnetic field strength, $G^{\mu\nu}$ is the gluon field strength and v is the 4-velocity of the heavy meson. We set the light current quark masses to zero for our computation, since we are not interested in $O(e^2 m_{u,d,s})$ effects. The coefficient of the term $\overline{Q} D^2 Q$ is fixed by reparameterization

invariance [11]; the coefficients a and b can be fixed by matching in QCD perturbation theory and are unity to leading order in $\alpha_s(m_Q)$ [12].

All divergences that appear when computing physical quantities to $O(1/m_Q)$ can be absorbed into counterterms of the form appearing in eq. (2.5). This severely restricts the ultraviolet behavior of the matrix elements that appear in eq. (2.1), since only isospin-violating counterterms can contribute to the mass difference. Since we are neglecting light current quark masses, the only isospin-violating term in the lagrangian is the light-quark kinetic term $\bar{q}i\not{D}q$. (It violates isospin because the light quarks have different charges.) However, matrix elements of $\bar{q}i\not{D}q$ between physical states vanish by the equations of motion. Thus, for the $Q\bar{u}$–$Q\bar{d}$ mass difference, there is no counterterm contribution, and the integral on the right-hand side of eq. (2.1) must converge. (A similar argument in the context of full QCD is given in ref. [13].) In fact, since the quark charges are arbitrary parameters, the contributions from ΔT_{qq} and ΔT_{Qq} must individually converge.

The electromagnetic mass differences get contributions from isospin-violating counterterms at order $1/m_Q^2$, such as

$$\delta\mathcal{L} = \frac{c}{m_Q^2} F^{\mu\nu} \bar{q}\gamma_\mu D_\nu q. \tag{2.6}$$

The integral for ΔM is convergent in full QCD [13], so this simply means that the $O(1/m_Q^2)$ contribution to the electromagnetic mass difference is sensitive to momenta up to $\sim m_Q$, but the integrand falls off for momenta above m_Q fast enough so that the integral converges.

It is convenient to rewrite ΔT in a useful form using some elementary isospin group theory. We first split the the electromagnetic current into heavy-quark and light-quark components

$$J^\mu = J_Q^\mu + J_q^\mu. \tag{2.7}$$

The heavy-quark current is

$$J_Q^\mu = \mathcal{Q}_Q \overline{Q}\gamma^\mu Q, \tag{2.8}$$

where $\mathcal{Q}_c = \frac{2}{3}$, $\mathcal{Q}_b = -\frac{1}{3}$. The light-quark current can be decomposed into isospin 0 and 1 components

$$J_q^\mu = J_{q0}^\mu + J_{q1}^\mu, \tag{2.9}$$

where

$$J_{q0}^\mu = \frac{\mathcal{Q}_u + \mathcal{Q}_d}{2} \bar{q}\gamma^\mu q, \qquad J_{q1}^\mu = \frac{\mathcal{Q}_u - \mathcal{Q}_d}{2} \bar{q}\tau_3\gamma^\mu q, \qquad q \equiv \begin{pmatrix} u \\ d \end{pmatrix}, \tag{2.10}$$

and $\mathcal{Q}_u = \frac{2}{3}$, $\mathcal{Q}_d = -\frac{1}{3}$. We can then write

$$\Delta T = \Delta T_{qq} + \Delta T_{Qq} \tag{2.11}$$

in terms of the state $\langle P| \equiv \langle P_{Q\bar{u}}|$ alone:

$$\Delta T_{qq}(p,q) \equiv 2i \int d^4x\, e^{iq\cdot x} \langle P(p)| T J_{q0}^\mu(0) J_{q1\mu}(x) |P(p)\rangle + (q \to -q), \tag{2.12}$$

$$\Delta T_{Qq}(p,q) \equiv 2i \int d^4x\, e^{iq\cdot x} \langle P(p)| T J_Q^\mu(0) J_{q1\mu}(x) |P(p)\rangle + (q \to -q). \tag{2.13}$$

We denote the corresponding contributions to ΔM by ΔM_{qq} and ΔM_{Qq}, respectively.

Eq. (2.1) reduces the problem of computing the electromagnetic mass differences to the problem of determining the forward Compton amplitude ΔT defined in eq. (2.2). We address this problem by writing a dispersion relation that expresses ΔT in terms of measurable data. ΔT is a function of two kinematic invariants, which we take to be $\vec{q}^{\,2}$ and $\nu \equiv q_0$ in the frame where $p = (M_P, \vec{0})$. We can then write a spectral representation for ΔT by inserting a complete set of states into eq. (2.3). The result is

$$\Delta T(\nu, \vec{q}^{\,2}) = -\frac{2}{\pi} \int_0^\infty d\nu' \, \frac{\nu' \operatorname{Im} \Delta T(\nu', \vec{q}^{\,2})}{\nu^2 - \nu'^2 + i0+}, \tag{2.14}$$

which can be thought of as a fixed-$\vec{q}^{\,2}$ dispersion relation. Dispersion relations such as this in general require subtraction because of the ultraviolet behavior of products of currents. However, due to isospin invariance, there is no counterterm that modifies the product of currents appearing in ΔT, so the operation of inserting a complete set of states that was used to derive eq. (2.14) is valid without modification.*

$\operatorname{Im} \Delta T$ is determined by physical matrix elements. In the $m_Q \to \infty$ limit, the relation is particularly simple:

$$\operatorname{Im} \Delta T(\nu, \vec{q}^{\,2}) = 2\pi \sum_n \delta(\nu - \Delta_n) \langle P(p) | J_0^\mu(0) | n(\vec{q}) \rangle \langle n(\vec{q}) | J_{1\mu}(0) | P(p) \rangle + \text{h.c.}, \tag{2.15}$$

where $M + \Delta_n$ is the mass of the state n, and $J_0^\mu \equiv J_{q0}^\mu + J_Q^\mu$, $J_1^\mu \equiv J_{q1}^\mu$. The sum over n is over a complete set of states with fixed 3-momentum $\vec{q}$. This in principle solves the problem of determining ΔT in terms of physical data, since $\operatorname{Im} \Delta T(\nu, q^2)$ is determined by structure functions for inelastic scattering off the heavy meson. However, eqs. (2.14) and (2.15) by themselves do not give a practical method of determining ΔT; for example, experimentally inaccessible data for timelike photon momenta ("timelike structure functions") are required on the right-hand side of eq. (2.14). We will see in the next section that progress can still be made in the context of the large-N limit of QCD.

The main use of the dispersion relation eq. (2.14) for our purposes is that it makes manifest that the electromagnetic mass differences are insensitive to the properties of very heavy states. We have argued that the integral that determines ΔM converges, so it is dominated by $\Delta T(\nu, \vec{q}^{\,2})$ with $\nu^2 \sim \vec{q}^{\,2} \sim \Lambda_{\text{QCD}}^2$. Now consider the contributions of an intermediate state n to ΔT in this kinematic region: the delta function in eq. (2.15) means that a state with mass $M + \Delta_n$ contributes at $\nu' = \Delta_n$ in the right-hand side of eq. (2.14). Therefore, the fact that the integral over ν' in eq. (2.14) converges means that the contribution of states with $\Delta_n \gg \Lambda_{\text{QCD}}$ is suppressed.

The dispersion relation eq. (2.14) also shows that we can continue the integral in eq. (2.1) to Euclidean momenta. (Specifically, we write $q = (iq_{E0}, \vec{q})$, so that $d^4q = i d^4 q_E$ and $q^2 = -q_E^2$.) We

* In more detail: the operators $J_{0,1}^\mu$ are defined by functional differentiation with respect to appropriate sources $S_{0,1}^\mu$ added to the lagrangian. All divergences in this lagrangian can be canceled by counterterms depending on the sources with dimension 4 or less. Because there is no counterterm of the form $S_0 S_1$ allowed by isospin invariance, there is no short-distance singularity in the product of currents appearing in ΔT, and the manipulations used to derive eq. (2.14) are valid. It is interesting to note that it was guessed that eq. (2.14) did not require subtraction before the advent of QCD; see for example ref. [4].

also note that because the integrand is integrated over $d^4 q_E$, we can angularly average it in Euclidean momentum space without changing the result. We then have

$$\Delta M = \frac{e^2}{2} \int \frac{d^4 q_E}{(2\pi)^4} \frac{\langle \Delta T \rangle (q_E^2)}{q_E^2}, \tag{2.16}$$

where $\langle \Delta T \rangle$ is the angular average of ΔT. Eq. (2.16) shows that we need not determine the full kinematic behavior of ΔT. For example, $\langle \Delta T \rangle$ can be shown to be insensitive to the scaling behavior in the deep-inelastic region, in agreement with the fact that infinitely many states contribute to scaling behavior [5].

This problem of computing hadron electromagnetic mass differences has a long and involved history [4][5], and we will not give a detailed discussion of the literature. We do wish to remark that one common approach to the problem of baryon electromagnetic mass differences has been to write a fixed-q^2 (as opposed to fixed-$\vec{q}^{\,2}$) dispersion relation and attempt to express the electromagnetic mass differences in terms of measured *spacelike* structure functions [3][4][5]. However, there are good reasons to believe that the fixed-q^2 dispersion relation requires subtraction. (The necessity of the subtraction is related to the small x behavior of the structure functions; see *e.g.* ref. [5].) Since the electromagnetic splittings cannot be determined without knowing the subtraction constant, we do not follow this approach in this paper. We will, however, make some brief comments about the relation between this approach and ours in the next section.

3. ΔM in the Large-N and Heavy-quark Limit

In this section, we consider the computation of ΔM in the combined large-N and heavy-quark limit. We will consider $O(1/m_Q)$ corrections in the next section, but we will not attempt to go beyond leading order in the $1/N$ expansion in this paper.

For large N, QCD reduces to a weakly-coupled field theory with infinitely many meson fields whose interactions are polynomials in momenta [14]. The transition amplitudes can be expanded systematically in powers of $1/N$, and the leading term in this expansion corresponds to keeping only tree graphs in the mesonic theory. The meson graphs that contribute to ΔT are shown in fig. 1.

The representation of ΔT as a sum of tree graphs gives a kind of double-dispersion relation that will allow us to determine the heavy-meson electromagnetic mass differences in terms of measurable matrix elements. For example, it is clear that the graphs with intermediate heavy-meson lines (the first graph in fig. 1) are related to heavy meson form factors. However, we must also obtain information about the remaining graphs, which are not obviously related to form factors. We will do this by imposing consistency between the hadronic theory and properties of QCD and heavy-quark effective theory, such as those discussed in the previous section. These consistency conditions are expressed in terms of sum rules relating the (infinitely many) couplings of the states that appear in the graphs. If we assume that the sum rules are saturated by the lowest-lying states (consistent with the low-energy dominance proved above), we will find that the unknown contributions from the graphs in fig. 1 with no heavy-meson intermediate states give a numerically negligible contribution to the electromagnetic mass difference.

For baryons, meson loops are *not* suppressed in the large-N limit (see [14][15]), so the arguments in this section cannot be simply extended to this case. Recent progress in the $1/N$ expansion for baryons [15] may be relevant to overcoming this difficulty.

We now consider the contributions ΔM_{qq} and ΔM_{Qq} (defined by eqs. (2.12) and (2.13)) in turn.

3.1. ΔM_{qq}

Consider ΔM_{qq} from the point of view of the large-N mesonic field theory. The sum of graphs that gives the amplitude ΔT_{qq} can be written

$$\Delta T_{qq}(\nu^2, q^2) = -2 \sum_n \frac{\Delta_n W_n(\nu^2, q^2)}{\nu^2 - \Delta_n^2 + i0+} + 2C(\nu^2, q^2), \tag{3.1}$$

where the first term is the contribution from the first graph in fig. 1, and the second term is the sum of the remaining graphs. Because ΔT_{qq} is even under $q \mapsto -q$, we consider it a function of ν^2 rather than ν. The sum on n in the first term runs over the (infinitely many) heavy mesons with mass $M + \Delta_n$ that appear in the intermediate-state propagators in the first graph of fig. 1. The pole structure in this term arises by combining the heavy-meson propagators $1/(\pm\nu - \Delta_n + i0+)$. The vertices in the large-N limit are polynomials, and so the only non-polynomial dependence of W_n and C in eq. (3.1) comes from the meson propagators in figs. 1 and 2, which give poles in q^2. Therefore, W_n and C are polynomial in ν^2.

We can give a physical interpretation of $W_n(\nu^2, q^2)$ by noting that for q^2 spacelike, the only real intermediate states that can appear in ΔT_{qq} in the large-N limit are single heavy meson states. From eq. (3.1) we compute

$$\operatorname{Im} \Delta T_{qq}(\nu^2, q^2) = \pi \sum_n W_n(q^2) \delta(\nu - \Delta_n), \tag{3.2}$$

where we have defined $W_n(q^2) \equiv W_n(\Delta_n^2, q^2)$. Substituting this into the spectral representation eq. (2.15), we obtain

$$W_n(q^2) = 2 \langle P(p) | J^\mu_{q0}(0) | n(\vec{q}) \rangle \langle n(\vec{q}) | J_{q1\mu}(0) | P(p) \rangle + \text{h.c.} \tag{3.3}$$

Therefore, $W_n(q^2)$ is proportional to on-shell heavy-meson form factors. (We can also see this directly by looking at diagrams.)

Because the functions W_n are polynomial in ν^2, we can expand W_n around $\nu^2 = \Delta_n^2$ to obtain

$$\Delta T_{qq}(\nu^2, q^2) = -2 \sum_n \left[\frac{\Delta_n W_n(q^2)}{\nu^2 - \Delta_n^2 + i0+} - D_n(\nu^2, q^2) \right] + 2C(\nu^2, q^2), \tag{3.4}$$

where $D_n(\nu^2, q^2)$ is polynomial in ν^2. The terms in eq. (3.4) proportional to $W_n(q^2)$ depend only on form factors, and we will refer to the contribution of these terms as the "form-factor" contribution. We refer to the remaining terms as "contact" contributions. (The first term in the sum over n is exactly what one would write for the right-hand side of an *unsubtracted* fixed-q^2 dispersion relation. As mentioned in section 2, the fixed-q^2 dispersion relation requires subtraction, and so we expect the two terms in the sum on n to separately diverge, while the total sum is finite. For this reason, we do not combine $\sum_n D_n$ with C.)

The electromagnetic mass difference can therefore be written

$$\Delta M_{qq} = e^2 \int \frac{d^4 q_E}{(2\pi)^4} \frac{1}{q_E^2} \left\{ \sum_n \left[\frac{\Delta_n W_n(-q_E^2)}{(q_{E4})^2 + \Delta_n^2} + \langle D_n \rangle (q_E^2) \right] + \langle C \rangle (q_E^2) \right\}, \tag{3.5}$$

where we have continued the integral to Euclidean momentum space. (Recall that $\langle \cdot \rangle$ denotes the 4-dimensional angular average.)

In the large-N limit, the functions $C(\nu^2, q^2)$ and $D_n(\nu^2, q^2)$ have the form

$$C(\nu^2, q^2),\ D_n(\nu^2, q^2) \sim \sum \frac{\text{polynomial in } \nu^2,\, q^2}{\text{polynomial in } q^2}, \tag{3.6}$$

where the polynomials in the denominator come from the vector-meson propagators in fig. 1. Therefore, after angular averaging in Euclidean space, we have

$$\langle C\rangle(q_E^2) = \sum_{r,s} \left[\frac{C_{rs}^{(0)} + C_{rs}^{(1)} q_E^2}{(q_E^2 + m_{0r}^2)(q_E^2 + m_{1s}^2)} + P_{rs}^{(C)}(q_E^2) \right], \tag{3.7}$$

$$\langle D_n\rangle(q_E^2) = \sum_{r,s} \left[\frac{D_{nrs}^{(0)} + D_{nrs}^{(1)} q_E^2}{(q_E^2 + m_{0r}^2)(q_E^2 + m_{1s}^2)} + P_{nrs}^{(D)}(q_E^2) \right], \tag{3.8}$$

where the sum over r, s runs over the vector mesons that appear in the propagators in fig. 1; these have isospin 0 and 1, respectively. (The contributions from terms that do not have two vector-meson poles can be written in this form by choosing the polynomials in the numerator to cancel the pole factors appearing in the denominator.) In eqs. (3.7) and (3.8), $C^{(0)}, \ldots, D^{(1)}$ are constants, and $P^{(C)}$ and $P^{(D)}$ are polynomials in q_E^2.

The unknown quantities on the right-hand sides of eqs. (3.7) and (3.8) determine the contact contribution to the electromagnetic mass difference. Our strategy will be to restrict these quantities by imposing various consistency conditions arising from the definition of ΔT_{qq} in QCD. These consistency conditions will involve the contact contribution alone because the form factor contribution will satisfy the consistency conditions by itself.

We first note that the integral in eq. (3.5) must converge (as shown in section 2). It is expected on very general grounds that the form factors contributing to $W_n(q^2)$ fall off at large q^2 (see below), so that for each n the form-factor contribution in eq. (3.5) is convergent. On the other hand, for general values of the coefficients in eqs. (3.7) and (3.8), the contact contribution will diverge in the ultraviolet, so the condition that the integral for ΔM is finite will constrain the contact contribution. If we impose a cutoff Λ on the photon momentum (note that this is gauge invariant), the divergences then have the simple forms Λ^{2k} ($k = 1, 2, \ldots$) and $\log \Lambda^2$. Demanding that the coefficients of these divergences separately vanish gives the "ultraviolet" sum rules

$$\sum_{r,s} P_{rs}^{(C)}(q_E^2) + \sum_n \sum_{r,s} P_{nrs}^{(D)}(q_E^2) = 0, \tag{3.9}$$

$$\sum_{r,s} C_{rs}^{(1)} + \sum_n \sum_{r,s} D_{nrs}^{(1)} = 0. \tag{3.10}$$

Note that eq. (3.9) simply means that the polynomials $P^{(C)}$ and $P^{(D)}$ do not contribute to ΔM. To derive these sum rules, it is crucial that the meson vertices are polynomial in momenta, so that there is a maximum possible power divergence Λ^{2k}. If arbitrarily high powers of Λ^2 were possible, then we would not be able to unambiguously separate the logarithmic and power divergences, since an infinite series in Λ^2 may behave asymptotically like $\ln \Lambda^2$.*

* If the vertices that contribute to D_n are polynomials of higher degree as n becomes large, the integral in eq. (3.5) can give rise to arbitrarily high powers of Λ^2. However, if we make some reasonable assumptions about the asymptotic properties of correlation functions in QCD, this can be shown not to occur.

Next, we note that as $q \to 0$, the form factors simply measure the appropriate charge, so from eq. (3.3) we have

$$W_n(q^2) \to (\mathcal{Q}_u^2 - \mathcal{Q}_d^2)\delta_{nP} \quad \text{as } q \to 0. \tag{3.11}$$

Substituting into eq. (3.4), we obtain

$$\Delta T_{qq}(p,q) \to 2\pi i(\mathcal{Q}_u^2 - \mathcal{Q}_d^2)\delta(q_0) + 2\sum_n D_n(0,0) + 2C(0,0) \quad \text{as } q \to 0. \tag{3.12}$$

(In the first term, we have used the identity $1/(q_0 + i0+) + (q \to -q) = -2\pi i\delta(q_0)$ to rewrite the P propagator.) The first term by itself is the correct result as $q \to 0$, as can be seen by writing a low-energy effective theory in which only the P and the P^* appear. Taking the ultraviolet sum rule eq. (3.9) into account, we obtain the "infrared" sum rule

$$\sum_{r,s} \frac{C_{rs}^{(0)}}{m_{0r}^2 m_{1s}^2} + \sum_n \sum_{r,s} \frac{D_{nrs}^{(0)}}{m_{0r}^2 m_{1s}^2} = 0. \tag{3.13}$$

For the expert, we note that the sum rules in eqs. (3.10) and (3.13) are exact analogs of the Weinberg sum rules used in the calculation of the π^+–π^0 mass difference, in the sense that the Weinberg sum rules can be derived from identical considerations applied to the vacuum correlation function that appears in that calculation [6].

So far, all of the approximations we have made have been controlled, *i.e.* they become arbitrarily accurate as some parameters of the underlying theory approach limiting values ($N \gg 1$ and $m_Q \gg \Lambda_{\text{QCD}}$). The result in eq. (3.5) is an infinite sum over one-particle states with infinitely many unknown parameters, even after the sum rules have been imposed. However, we know from the fixed-$\vec{q}^{\,2}$ dispersion relation discussed in section 2 that intermediate states with large mass do not contribute significantly to ΔM. We do not know how many terms in eq. (3.5) are needed to get a good approximation to the sum, since there is no known small parameter controlling the convergence of the series.

In order to make progress, we simply *assume* that the sum is well approximated by the first nontrivial term, *i.e.* that it is a good approximation to retaining only the mimimal set of intermediate states that gives a consistent description of the matrix elements that appear in the calculation. In particular, it is crucial that the matrix elements have the correct ultraviolet and infrared behavior to satisfy the sum rules derived above.

Clearly, we must include both the P and the P^* as intermediate states, since they become degenerate in the heavy-quark limit. To see what other states we must include, we consider the matrix elements

$$\langle P(p)|J^\mu_{q0,1}(0)|P(p+q)\rangle = F_{0,1}(-q^2)\, v^\mu, \tag{3.14}$$

$$\langle P(p)|J^\mu_{q0,1}(0)|P^*(p+q,\epsilon)\rangle = iG_{0,1}(-q^2)\, \epsilon^{\mu\nu\rho\sigma} v_\nu q_\rho \epsilon_\sigma, \tag{3.15}$$

which determine the form factor contribution if we include no heavy mesons other than the P and P^*. In the appendix, we use the constituent counting rules [16] to show that the form factors F and G fall off as $\sim 1/q^4$ for large spacelike momentum transfer q. In the large-N limit, the falloff of these form factors is due to the presence of single vector-meson intermediate states (see fig. 2). Since a single vector-meson pole can fall off at most as $\sim 1/q^2$, we see that the asymptotic behavior

requires cancelation between at least two different vector-meson intermediate states. The minimal set of vector-meson states we must consider therefore consists of the lightest isospin-1 vector mesons ρ and ρ' and the corresponding isospin-0 mesons ω and ω'. From the behavior of the form factors at small and large q, we can write

$$F_{0,1}(q_E^2) \simeq -\frac{\mathcal{Q}_u \pm \mathcal{Q}_d}{2} \frac{1}{(1 + q_E^2/m_\rho^2)(1 + q_E^2/m_{\rho'}^2)}, \tag{3.16}$$

$$G_{0,1}(q_E^2) \simeq -\frac{\mathcal{Q}_u \pm \mathcal{Q}_d}{2} \frac{\beta/2}{(1 + q_E^2/m_\rho^2)(1 + q_E^2/m_{\rho'}^2)}, \tag{3.17}$$

where we have used the fact that $m_\rho = m_\omega$ and $m_{\rho'} = m_{\omega'}$ in the large-N limit. (Numerically, $m_\omega - m_\rho = 14$ MeV.) Here, β is a strong-interaction matrix element that can be determined from the $P^* \to P\gamma$ decay rate. It is important to note that eqs. (3.16) and (3.17) are to be understood as valid only for the small q_E^2 that dominate in the integral that determines ΔM.

We therefore truncate the intermediate states by keeping only the contributions from the P, P^*, ρ, ω, ρ', and ω'. The contribution to ΔM_{qq} from the form factor contribution to eq. (3.5) can be directly evaluated using eqs. (3.3) and (3.14)–(3.17). We have used the language of dispersion theory to make the connection to physical matrix elements explicit, but our final result can be stated very simply in terms of Feynman graphs: we evaluate the one-loop electromagnetic self-energy for the heavy meson with momentum-dependent photon couplings as given in eqs. (3.16) and (3.17). The result is

$$\Delta M_{qq}^{\text{form factor}} \simeq (\mathcal{Q}_u^2 - \mathcal{Q}_d^2)\alpha \left[\frac{m_\rho}{4} \frac{1 + 3x + x^2}{(1+x)^3} - \frac{\beta^2 m_\rho^3}{8} \frac{1}{(1+x)^3} \right], \tag{3.18}$$

where

$$x \equiv \frac{m_\rho}{m_{\rho'}} = 0.53. \tag{3.19}$$

(We use the symbol "$\simeq$" to denote statements that are valid only with the truncation of states discussed above.) Numerically,

$$\Delta M_{qq}^{\text{form factor}} \simeq +0.38 - 0.039 \left(\frac{\beta}{1 \text{ GeV}^{-1}} \right)^2 \text{ MeV}. \tag{3.20}$$

The value of β has not yet been measured directly. (There is a weak upper bound that results from using $SU(3)$ symmetry applied to the decays $D^* \to D\gamma$ and $D^* \to D\pi$ [17]. Hadronic models tend to give values of $D^* \to D\gamma$ and $D^* \to D\pi$ consistent with β near 1 GeV^{-1} [18].) We will see that the contribution in eq. (3.20) is numerically small compared to ΔM_{Qq} computed in the next subsection, and the uncertainty in the value of β will not be very important for our final results. This issue will be analyzed in greater detail in a subsequent paper [8].

We now turn to the contact contribution in eq. (3.4). The ultraviolet sum rule eq. (3.10) allows us to parameterize the contact contribution as

$$\langle \Delta T^{\text{contact}} \rangle (q_E^2) \simeq (\mathcal{Q}_u^2 - \mathcal{Q}_d^2) m_\rho^3 \left[\frac{A_{\rho\rho}}{(q_E^2 + m_\rho^2)^2} + \frac{A_{\rho\rho'}}{(q_E^2 + m_\rho^2)(q_E^2 + m_{\rho'}^2)} + \frac{A_{\rho'\rho'}}{(q_E^2 + m_{\rho'}^2)^2} \right], \tag{3.21}$$

where the A's are linear combinations of the coefficients C and D defined in eq. (3.5). We have normalized the A's so that they are ~ 1 if we identify m_ρ with the hadronic scale that appears in these matrix elements. Imposing the infrared sum rule eq. (3.13), we obtain

$$A_{\rho\rho} \simeq -x^2 A_{\rho\rho'} - x^4 A_{\rho'\rho'}. \tag{3.22}$$

It is now straightforward to compute the contact contribution to the electromagnetic mass difference. We obtain

$$\Delta M_{qq}^{\rm contact} \simeq -(\mathcal{Q}_u^2 - \mathcal{Q}_d^2)\frac{\alpha m_\rho}{8\pi} x^2 \left[A_{\rho\rho'} \frac{1 - x^2 + \ln x^2}{1 - x^2} + A_{\rho'\rho'}(1 - x^2) \right] \tag{3.23}$$

$$= -0.02\, A_{\rho\rho'} + 0.02\, A_{\rho'\rho'} \text{ MeV}. \tag{3.24}$$

We see that for any reasonable value of the A's, this contribution is numerically negligible, and ΔM_{qq} is given by eq. (3.20) to a good approximation.

Heavy-quark symmetry equates the electromagnetic mass difference for the P to that of the P^*. We have (in an obvious notation)

$$\Delta M_{qq}^* = \Delta M_{qq}. \tag{3.25}$$

Before continuing the discussion, we note that our final result is rather similar to the prescription given long ago for computing hadron electromagnetic mass differences [1], which was essentially to evaluate the electromagnetic self-energy with the photon vertices replaced by momentum-dependent form factors. However, we emphasize that this prescription was never justified in any satisfactory way. In particular, the form factors appearing used in this prescription must be evaluated off shell in order to compute the electromagnetic mass difference. Clearly, there are infinitely many ways to continue a form factor off shell, and these will give different final results. (In our formalism, this ambiguity is contained in the term $\langle D_n \rangle$ defined in eq. (3.5).) Also, neglecting the contact contributions (contained in C in our formalism) was never justified.

The use of the large-N limit and the truncation of intermediate states may seem like rather drastic approximations. However, we note that precisely analogous approximations are used in the classic result [2] for the π^+–π^0 mass difference: the π^+–π^0 mass difference is written in terms of vacuum correlation functions, which are then saturated with the appropriate lowest-lying 1-particle states in the large-N limit (in this case, the ρ and the a_1 vector mesons). In this way, one can obtain a formula similar to the ones obtained here for the heavy meson electromagnetic mass differences [2][6]:

$$m_{\pi^+} - m_{\pi^0} \simeq \frac{3\alpha}{8\pi m_\pi} \frac{m_\rho^2}{1 - m_\rho^2/m_a^2} \ln \frac{m_a^2}{m_\rho^2} \simeq 5.9 \text{ MeV}. \tag{3.26}$$

This is within 30% of the experimental value 4.6 MeV. We therefore adopt this as an estimate of the size of the error from the *combined* approximation of the large-N limit and the truncation of intermediate states.

As stated in the introduction, the discussion that follows consists mainly of repeated applications of the principles described above to compute the remaining contributions to ΔM. The weary reader interested mainly in the bottom line is invited to skip to section 5, which summarizes the main results and gives our conclusions.

3.2. ΔM_{Qq}

The computation of ΔM_{Qq} can be carried out following the same arguments given in subsection 3.1, so we will be brief. We can write

$$\Delta T_{Qq}(\nu^2, q^2) = -2 \sum_n \left[\frac{\Delta_n W_n(q^2)}{\nu^2 - \Delta_n^2 + i0+} - D_n(\nu^2, q^2) \right] + 2C(\nu^2, q^2), \tag{3.27}$$

where the functions W_n, D_n, and C are defined as in eq. (3.4), but with J_Q^μ replacing J_{q0}^μ.

The main difference between ΔT_{Qq} and ΔT_{qq} arises because the form factors of J_Q^μ are constants in the heavy-quark limit [19]:

$$\langle P(p)|J_Q^\mu(0)|P(p+q)\rangle = \mathcal{Q}_Q v^\mu, \tag{3.28}$$

$$\langle P(p)|J_Q^\mu(0)|P^*(p+q,\epsilon)\rangle = 0. \tag{3.29}$$

The fact that the form factors are constants even for large momenta can also be understood from the point of view of the constituent counting rules [16], since graphs in which the "hard" momentum flows directly into the heavy-quark line do not fall off at large $|q^2|$ (see the appendix). From the point of view of the large-N meson theory, we can understand this in the following way: the intermediate meson states that contribute to the form factor for J_Q^μ in the large-N limit are $Q\overline{Q}$ states, and so have mass of order $2m_Q$. These states are integrated out of the effective theory , and their effects are correctly taken into account by local interactions. The form factors are therefore pure polynomials in q^2 and ν in the heavy-quark limit. Because we do not expect the form factors to grow, they must be constants, and the constants are fixed by the value of the form factors at $q = 0$. (Similar arguments are made in ref. [20].)

From eq. (3.28), we then have

$$\langle C\rangle(q_E^2) = \sum_s \left[\frac{C_s^{(0)}}{q_E^2 + m_{1s}^2} + P_s^{(C)}(q_E^2)\right], \tag{3.30}$$

$$\langle D_n\rangle(q_E^2) = \sum_s \left[\frac{D_{ns}^{(0)}}{q_E^2 + m_{1s}^2} + P_{ns}^{(D)}(q_E^2)\right], \tag{3.31}$$

where the sum over s is over isospin-1 vector mesons that appear in the propagators in figs. 1 and 2. As before, $C_s^{(0)}$ and $D_{ns}^{(0)}$ are constants, while $P^{(C)}$ and $P^{(D)}$ are polynomials in q_E^2.

We now write ultraviolet and infrared sum rules using the same reasoning used in the previous subsection. We have

$$\sum_s P_s^{(C)}(q_E^2) + \sum_n \sum_s P_{ns}^{(D)}(q_E^2) = 0, \tag{3.32}$$

$$\sum_s C_s^{(0)} + \sum_n \sum_s D_{ns}^{(0)} = 0, \tag{3.33}$$

$$\sum_s \frac{C_s^{(0)}}{m_{1s}^2} + \sum_n \sum_s \frac{D_{ns}^{(0)}}{m_{1s}^2} = 0. \tag{3.34}$$

As before, we approximate ΔM_{Qq} by keeping only the contributions of the lowest-lying states; in this case, only the P heavy meson state and the ρ and ρ' vector mesons are required.

It is not hard to see that the contact contribution is forced to vanish identically when the sum rules are saturated by ρ and ρ' vector mesons. The form factor for the light-quark current is given in eq. (3.16) in the approximation we are making. Substituting this and the heavy-quark current form factor in eq. (3.28) into an expression for ΔM_{Qq} analogous to eq. (3.5), we obtain

$$\Delta M_{Qq} \simeq -\mathcal{Q}_Q(\mathcal{Q}_u - \mathcal{Q}_d)\frac{\alpha m_\rho}{1+x} \tag{3.35}$$

$$= \begin{cases} -2.5 \text{ MeV} & \text{for } Q = c, \\ +1.2 \text{ MeV} & \text{for } Q = b. \end{cases} \tag{3.36}$$

We see that this contribution numerically dominates the light-quark current contribution of eq. (3.20). Once again, heavy-quark symmetry gives

$$\Delta M^*_{Qq} = \Delta M_{Qq}. \tag{3.37}$$

4. $1/m_Q$ Corrections

In this section, we consider the $1/m_Q$ corrections to the results obtained above, still working in the large-N limit. For mesons containing a c quark, these corrections are expected to be substantial, and so it is important to estimate them.

4.1. ΔM_{qq}

To include $O(1/m_Q)$ effects, we must make two types of changes to the formulas of the previous section. First, the meson vertices and the masses change by $O(1/m_Q)$ effects. Second, the form of the heavy-meson propagators is modified. This modification can be thought of as a recoil correction: it is completely kinematical in origin, and hence determined with no dynamical input. Formally, this can be thought of as writing a non-relativistic expansion of the fully relativistic propagator or (more correctly) an expression of reparameterization invariance [11]. The result is that eq. (3.5) in the previous section is replaced with

$$\Delta M_{qq} = e^2 \int \frac{d^4 q_E}{(2\pi)^4} \frac{1}{q_E^2} \left\{ \sum_n \left[\frac{(\Delta_n + \delta\Delta_n(q_E)) W_n(-q_E^2)}{(q_{E4})^2 + \Delta_n^2} + \langle D_n \rangle (q_E^2) \right] + \langle C \rangle (q_E^2) \right\}, \tag{4.1}$$

where

$$\delta\Delta_n(q_E) = \frac{q_E^2 + \Delta_n^2}{2M} \left(1 - \frac{2\Delta_n^2}{q_{E4}^2 + \Delta_n^2} \right). \tag{4.2}$$

The functions $\langle C \rangle$ and $\langle D_n \rangle$ are given by formulas identical to eqs. (3.7) and (3.8), except that the coefficients are now given in a power series in $1/m_Q$:

$$\begin{aligned} C_{rs}^{(0)} &= C_{rs}^{(0,0)} + C_{rs}^{(0,1)} + O(1/m_Q^2), \\ \Delta M_{qq} &= \Delta M_{qq}^{(0)} + \Delta M_{qq}^{(1)} + O(1/m_Q^2), \end{aligned} \tag{4.3}$$

etc., where $C^{(0,k)}$, $\Delta M_{qq}^{(k)}, \ldots$ parameterize the $O(1/m_Q^k)$ corrections.

We obtain ultraviolet sum rules by imposing the condition that the integral for ΔM_{qq} converges order by order in $1/m_Q$. Because the form factors fall off as $\sim 1/q^3$ (see the appendix), the form-factor contribution for each n converges by itself, and the sum rules again constrain the contact contribution alone:

$$\sum_{r,s} P_{rs}^{(C)}(q_E^2) + \sum_n \sum_{r,s} P_{nrs}^{(D)}(q_E^2) = 0, \tag{4.4}$$

$$\sum_{r,s} C_{rs}^{(1,j)} + \sum_n \sum_{r,s} D_{nrs}^{(1,j)} = 0, \quad j = 0, 1. \tag{4.5}$$

To get the infrared sum rules, we note that the recoil corrections do not affect the leading behavior of the integrand as $q_E \to 0$. However, there is a new feature that enters at $O(1/m_Q)$ because

$$\Delta T(p,q) \to (\mathcal{Q}_u^2 - \mathcal{Q}_d^2) \left[2\pi i \delta(q_0) + \frac{4}{M} \right] \quad \text{as } q \to 0. \tag{4.6}$$

This results by writing a low-energy effective theory where only the P and the P^* appear. In this theory, the "extra" $1/M$ term arises from the term

$$\delta\mathcal{L}_{\text{eff}} = -\frac{1}{2M} P^\dagger D^2 P, \tag{4.7}$$

where the coefficient is fixed exactly by reparameterization invariance. (Other interactions that arise at order $1/m_Q$ vanish as $q \to 0$.) Because of this, the infrared sum rules are

$$\sum_{r,s} \frac{C_{rs}^{(0,0)}}{m_{0r}^2 m_{1s}^2} + \sum_n \sum_{r,s} \frac{D_{nrs}^{(0,0)}}{m_{0r}^2 m_{1s}^2} = 0. \tag{4.8}$$

$$\sum_{r,s} \frac{C_{rs}^{(0,1)}}{m_{0r}^2 m_{1s}^2} + \sum_n \sum_{r,s} \frac{D_{nrs}^{(0,1)}}{m_{0r}^2 m_{1s}^2} = \frac{2}{M}(\mathcal{Q}_u^2 - \mathcal{Q}_d^2) \tag{4.9}$$

We have once again used the fact that the form factor contribution coming from $W_n(q^2)$ already has the correct infrared behavior, except for the term in eq. (4.7).

These sum rules can be saturated by the same set of states as in the previous section: ρ, ω, ρ', ω', P, and P^*. The form factor contribution is convergent by itself, and is given by

$$\begin{aligned}\Delta M_{qq}^{(1)\,\text{form factor}} \simeq {} & -\frac{(\mathcal{Q}_u^2 - \mathcal{Q}_d^2)\alpha m_\rho^2}{4\pi M} \frac{1 + 2x^2 \ln x^2 - x^4}{(1-x^2)^3} \\ & + \frac{3(\mathcal{Q}_u^2 - \mathcal{Q}_d^2)\alpha\beta^2 m_\rho^2}{8\pi} \left[(M^* - M) \frac{1 + 2x^2 \ln x^2 - x^4}{(1-x^2)^3} \right. \\ & \left. \qquad - \frac{m_\rho^2}{2M} \frac{2(1-x^2) + (1+x^2)\ln x^2}{(1-x^2)^3} \right].\end{aligned} \tag{4.10}$$

Using the infrared sum rule, the contact contribution can be written (compare to eq. (3.21))

$$\begin{aligned}\langle \Delta T^{(1)\,\text{contact}} \rangle (q_E^2) \simeq {} & \frac{4}{M} \frac{(\mathcal{Q}_u^2 - \mathcal{Q}_d^2) m_\rho^4}{(q_E^2 + m_\rho^2)^2} \\ & + (\mathcal{Q}_u^2 - \mathcal{Q}_d^2) m_\rho^3 \left[\frac{A_{\rho\rho}^{(1)}}{(q_E^2 + m_\rho^2)^2} + \frac{A_{\rho\rho'}^{(1)}}{(q_E^2 + m_\rho^2)(q_E^2 + m_{\rho'}^2)} + \frac{A_{\rho'\rho'}^{(1)}}{(q_E^2 + m_{\rho'}^2)^2} \right].\end{aligned} \tag{4.11}$$

The constants $A^{(1)} \sim m_\rho / M$ parameterize the $O(1/m_Q)$ corrections to the unknown contact contributions defined in eq. (3.21). The ultraviolet sum rule then gives

$$A_{\rho\rho}^{(1)} \simeq -x^2 A_{\rho\rho'}^{(1)} - x^4 A_{\rho'\rho'}^{(1)}, \tag{4.12}$$

and we obtain

$$\begin{aligned}\Delta M_{qq}^{(1)\,\text{contact}} \simeq {} & \frac{(\mathcal{Q}_u^2 - \mathcal{Q}_d^2)\alpha m_\rho}{8\pi M} \\ & - \frac{(\mathcal{Q}_u^2 - \mathcal{Q}_d^2)\alpha m_\rho}{8\pi} x^2 \left[A_{\rho\rho'}^{(1)} \frac{1 - x^2 + \ln x^2}{1 - x^2} + A_{\rho'\rho'}^{(1)} (1 - x^2) \right].\end{aligned} \tag{4.13}$$

We find that the corrections that depend on the A's are numerically negligible for reasonable values of $A^{(1)}$. Our result is therefore

$$\Delta M_{qq}^{(1)} \simeq \begin{cases} 0.088 + 0.028 \left(\dfrac{\beta}{1\ \text{GeV}^{-1}} \right)^2 \ \text{MeV} & \text{for } Q = c, \\[2ex] 0.031 + 0.0094 \left(\dfrac{\beta}{1\ \text{GeV}^{-1}} \right)^2 \ \text{MeV} & \text{for } Q = b. \end{cases} \tag{4.14}$$

Comparing to the lowest-order results in eq. (3.20), we see that the $1/m_Q$ corrections are 35% for the c system and 10% for the b system for $\beta = 1\ \mathrm{GeV}^{-1}$.

At order $1/m_Q$ the P^* electromagnetic mass difference is no longer equal to that of the P. Applying the same method, we obtain the isospin-violating hyperfine splitting due to the light quark charges,

$$\begin{aligned}\Delta M^*_{qq} - \Delta M_{qq} \simeq &-\frac{(\mathcal{Q}_u^2 - \mathcal{Q}_d^2)\alpha\beta^2 m_\rho^2}{2\pi}\,(M^* - M)\,\frac{1 + 2x^2\ln x^2 - x^4}{(1-x^2)^3}\\ &-\frac{(\mathcal{Q}_u^2 - \mathcal{Q}_d^2)\alpha\beta m_\rho^3}{6}\,(\beta' - \beta)\,\frac{1}{(1+x)^3}\end{aligned} \tag{4.15}$$

where β' measures the strength of a P^*–P^*–γ coupling. Heavy-quark symmetry gives $\beta' = \beta + O(1/m_Q)$, but we do not have any experimental information about the difference $\beta' - \beta$. We therefore obtain

$$\Delta M^*_{qq} - \Delta M_{qq} \simeq \begin{cases} -0.16 - 0.015\left(\dfrac{\beta}{1\ \mathrm{GeV}^{-1}}\right)^2\left(\dfrac{\beta'-\beta}{0.3\ \beta}\right)\ \mathrm{MeV} & \text{for } Q = c,\\[2ex] -0.054 - 0.0052\left(\dfrac{\beta}{1\ \mathrm{GeV}^{-1}}\right)^2\left(\dfrac{\beta'-\beta}{0.1\ \beta}\right)\ \mathrm{MeV} & \text{for } Q = b,\end{cases} \tag{4.16}$$

where we have normalized the value of $\beta' - \beta$ to a representative value. Comparing to eq. (3.20), we see that the $1/m_Q$ correction to ΔM^*_{qq} is 20% for the c system and 5% for the b system for the values of β and β' used to normalize these expressions.

4.2. ΔM_{Qq}

When we include the $O(1/m_Q)$ corrections, ΔM_{Qq} is given by a formula just like eq. (4.1), but with J_Q^μ replacing J_{q0}^μ. The functions $\langle C\rangle$ and $\langle D_n\rangle$ are given by formulas like eqs. (3.30) and (3.31), but with the coefficients expanded in powers of $1/m_Q$, as in the last section (see eq. (4.3)). Using reasoning similar to that of the previous subsection, we obtain ultraviolet sum rules of the same form as eqs. (4.4) and (4.5), and infrared sum rules of the same form as eqs. (4.8) and (4.9). The contact contribution is uniquely determined when the sum rules are imposed. Performing the necessary computations, we obtain

$$\Delta M^{(1)}_{Qq} \simeq \frac{\mathcal{Q}_Q(\mathcal{Q}_u - \mathcal{Q}_d)\alpha m_\rho^2}{2\pi M}\,\frac{\ln x^2}{1-x^2} + \frac{\mathcal{Q}_Q(\mathcal{Q}_u - \mathcal{Q}_d)\alpha\beta\beta_Q m_\rho^3}{2}\,\frac{1}{x(1+x)}, \tag{4.17}$$

where β_Q is the coupling of the heavy-quark current to P and P^*, normalized like β in eq. (3.17). In the heavy-quark limit,

$$\beta_Q = \frac{1}{M}\ [1 + O(\alpha_s(M))]\,. \tag{4.18}$$

The corrections are expected to be of order $\alpha_s(m_Q)/\pi$, which is 0.15 for the c quark and 0.08 for the b quark. Neglecting these corrections, we obtain

$$\Delta M^{(1)}_{Qq} = \begin{cases} -0.44 + 0.73\left(\dfrac{\beta}{1\ \mathrm{GeV}^{-1}}\right)\ \mathrm{MeV} & \text{for } Q = c,\\[2ex] +0.077 - 0.13\left(\dfrac{\beta}{1\ \mathrm{GeV}^{-1}}\right)\ \mathrm{MeV} & \text{for } Q = b.\end{cases} \tag{4.19}$$

Comparing to eq. (3.36), we see that this is a 10% correction for the c system and a 15% correction for the b system for $\beta = 1\ \mathrm{GeV}^{-1}$. (The small size of the corrections for the c system is a result of cancelations that depend on the value of β.)

Similarly, we can compute the P^* isospin splittings:

$$\Delta M^*_{Qq} - \Delta M_{Qq} \simeq -\frac{2\mathcal{Q}_Q(\mathcal{Q}_u - \mathcal{Q}_d)\alpha\beta\beta_Q m_\rho^3}{3}\frac{1}{x(1+x)}, \tag{4.20}$$

which gives (again using eq. (4.18))

$$\Delta M^*_{Qq} - \Delta M_{Qq} \simeq \begin{cases} -0.99\left(\dfrac{\beta}{1\ \mathrm{GeV}^{-1}}\right)\ \mathrm{MeV} & \text{for } Q = c, \\ +0.17\left(\dfrac{\beta}{1\ \mathrm{GeV}^{-1}}\right)\ \mathrm{MeV} & \text{for } Q = b. \end{cases} \tag{4.21}$$

5. Summary and Conclusions

We now summarize our results. We have computed the heavy-meson electromagnetic mass differences by working in the large-N limit, where the electromagnetic mass differences are given by a convergent sum over an infinite number of 1-particle intermediate states. We obtained sum rules relating the matrix elements appearing in this sum that enforce the correct ultraviolet and infrared behavior on the electromagnetic amplitudes that appear in the calculation. All of this is a rigorous consequence of QCD (in the large-N limit); however, in order to get numerical results, we truncated the infinite sum by keeping the smallest number of intermediate states that are capable of giving a consistent description of the matrix elements which appear in the sum. We argued that these approximations are similar to the ones made in the classic calculation of the π^+–π^0 mass difference, which works to 30%. Making these approximations, we find that the numerically dominant contribution comes from the heavy-quark current, and we obtain (see eqs. (3.20), (3.36), (4.14), and (4.19))

$$\begin{aligned}(M_{D^0} - M_{D^+})^{\mathrm{EM}} &\simeq \left[-2.4 + 0.74\left(\frac{\beta}{1\ \mathrm{GeV}^{-1}}\right) - 0.012\left(\frac{\beta}{1\ \mathrm{GeV}^{-1}}\right)^2\right]\ \mathrm{MeV} \\ &\quad + O(1/m_c^2), \end{aligned} \tag{5.1}$$

$$\begin{aligned}(M_{B^+} - M_{B^0})^{\mathrm{EM}} &\simeq \left[+1.7 - 0.13\left(\frac{\beta}{1\ \mathrm{GeV}^{-1}}\right) - 0.03\left(\frac{\beta}{1\ \mathrm{GeV}^{-1}}\right)^2\right]\ \mathrm{MeV} \\ &\quad + O(1/m_b^2). \end{aligned} \tag{5.2}$$

Here, β is a matrix element that measures the strength of the P^*–P–γ coupling (see eqs. (3.15) and (3.17)). The coefficients of the terms linear in β are $O(1/m_Q)$ and have perturbative QCD corrections of order $\alpha_{\mathrm{s}}(m_Q)$ (see eq. (4.18)). Similar results are also obtained for the vector mesons. Heavy quark symmetry gives $\Delta M^* = \Delta M + O(1/m_Q)$, and we obtain (see eqs. (4.16) and (4.21))

$$\begin{aligned}(M_{D^{*0}} - M_{D^{*+}})^{\mathrm{EM}} &\simeq (M_{D^0} - M_{D^+})^{\mathrm{EM}} \\ &\quad + \left[-0.16 - 0.99\left(\frac{\beta}{1\ \mathrm{GeV}^{-1}}\right) - 0.015\left(\frac{\beta}{1\ \mathrm{GeV}^{-1}}\right)^2\left(\frac{\beta' - \beta}{0.3\ \beta}\right)\right]\ \mathrm{MeV} \\ &\quad + O(1/m_c^2) \end{aligned} \tag{5.3}$$

$$\begin{aligned}(M_{B^{*+}} - M_{B^{*0}})^{\mathrm{EM}} &\simeq (M_{B^+} - M_{B^0})^{\mathrm{EM}} \\ &\quad + \left[-0.054 + 0.17\left(\frac{\beta}{1\ \mathrm{GeV}^{-1}}\right) - 0.0052\left(\frac{\beta}{1\ \mathrm{GeV}^{-1}}\right)^2\left(\frac{\beta' - \beta}{0.1\ \beta}\right)\right]\ \mathrm{MeV} \\ &\quad + O(1/m_b^2), \end{aligned} \tag{5.4}$$

where $\beta' = \beta + O(1/m_Q)$ is another unmeasured matrix element (see the discussion above eq. (4.16)). As above, the coefficients of the terms linear in β are $O(1/m_Q)$ and have corrections of order $\alpha_{\mathrm{s}}(m_Q)$.

For β near 1 GeV^{-1}, the heavy-quark expansion appears to be working very well for the b system, and moderately well for the c system. We believe that this is sufficiently encouraging to consider the phenomenology of these results in detail in a subsequent paper [8]. For the present, we hope that the systematic approach taken in this paper is the starting point for further progress on this classic problem in hadronic physics.

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Appendix A. Ultraviolet Behavior of Heavy-Meson Form Factors

In this appendix, we consider the ultraviolet behavior of matrix elements of the form

$$\langle P(p) | J_q^\mu(0) | n(p+q) \rangle, \tag{A.1}$$

where n is a heavy-light meson. We work in the the $1/m_Q$ expansion. This means that we consider spacelike momenta q in the limit

$$\Lambda_{\rm QCD}^2 \ll |q^2| \ll m_Q^2. \tag{A.2}$$

The ultraviolet behavior of these form factors can be determined from the constituent counting rules [16]. The idea is that for large spacelike momentum transfer, the leading behavior of the form factor can be obtained by power counting the hard momentum flow through constituent Feynman diagrams with appropriate kinematics for the initial and final state constituents. For the case under consideration, the leading graphs come from a single hard gluon exchange, as shown in the first two graphs in fig. 3. Evaluated in the heavy-quark effective theory, these graphs give a contribution of the form

$$\text{fig. 3} \sim \left\langle \frac{\overline{\psi}'(k') \not{v} (\not{k} + \not{q}) \gamma^\mu \psi(k)}{(k+q)^2 (xq)^2} + \frac{\overline{\psi}'(k') \gamma^\mu (\not{k} - x\not{q}) \not{v} \psi(k)}{(k-xq)^2 (xq)^2} \right\rangle + \cdots, \tag{A.3}$$

where x is the fraction of the hard momentum q flowing into the heavy quark. The average is taken over k and x. The dominant contribution will come from regions of integration where almost all of the hard momentum flows into the heavy quark (*i.e.*, $k \sim \Lambda_{\rm QCD}$ and $x = 1 + O(\Lambda_{\rm QCD}/m_Q)$). The reason is simply that if hard momentum flows into the light quark, the final state of the hard scattering subprocess consists of the heavy quark at rest and a light quark with momentum much larger than $\Lambda_{\rm QCD}$; such a state is expected to have very small overlap with a heavy meson at rest. Therefore,

$$\text{fig. 3} \sim \frac{q_\nu}{q^4} \left\langle \overline{\psi}'(k') \left[\not{v} \gamma^\nu \gamma^\mu - \gamma^\mu \gamma^\nu \not{v} \right] \psi(k) \right\rangle + \frac{1}{q^4} \left\langle \overline{\psi}'(k') \left[\not{v} \not{k} \gamma^\mu + \gamma^\mu \not{k} \not{v} \right] \psi(k) \right\rangle + \cdots, \tag{A.4}$$

and we have

$$\langle P(p) | J_q^\mu(0) | P(p+q) \rangle \sim \frac{\Lambda_{\rm QCD}^2}{q^4}, \tag{A.5}$$

$$\langle P(p) | J_q^\mu(0) | P^*(p+q) \rangle \sim \frac{\Lambda_{\rm QCD}}{q^3}, \tag{A.6}$$

etc. For the P elastic form factor, we have used the fact that rotational symmetry implies that the first term in eq. (A.4) is proportional to $q_\nu v^\mu v^\nu = O(1/m_Q)$, since the fact that the initial and final states have the same mass forces $q \cdot v = -q^2/2M$. In general, it is clear that matrix elements of the form of eq. (A.1) fall off at least as fast as $\sim 1/q^3$. This is the result quoted in the main text.

A complete analysis of the form factors would include a discussion of higher-order graphs such as the last one in fig. 3. We will not give a detailed analysis of this issue here. However, we expect graphs such as this to factorize, so that regions of integration where the gluon momenta are soft can be absorbed into "wavefunction" corrections, while the remaining hard contributions have the same asymptotic behavior as the contributions analyzed above (up to logarithms). For an example of this type of analysis, see ref. [21].

7. References

[1] See *e.g.* R. P. Feynman and G. Speisman, *Phys. Rev.* **94**, 500 (1954) ; M. Cini, E. Ferrari, and R. Gatto, *Phys. Rev. Lett.* **2**, 7 (1959) .

[2] T. Das, G. S. Guralnik, V. S. Mathur, F. E. Low, and J. E. Young, *Phys. Rev. Lett.* **18**, 759 (1967) ; T. Das, V. S. Mathur, S. Okubo, *Phys. Rev. Lett.* **19**, 859 (1967) .

[3] W. N. Cottingham, *Ann. Phys.* **25**, 424 (1963)

[4] R. P. Feynman, *Photon–Hadron Interactions* (Benjamin, 1972).

[5] A. Zee, *Phys. Rep.* **3**, 129 (1972) .

[6] S. Pokorski, *Gauge Field Theories* (Cambridge, 1987).

[7] J. L. Goity, *Phys. Lett.* **303B**, 337 (1993) .

[8] M. A. Luty and R. Sundrum, in preparation.

[9] S. Weinberg, *Phys. Rev. Lett.* **18**, 507 (1967) ; C. Bernard, A. Duncan, J. Lo Secco, and S. Weinberg, *Phys. Rev.* **D12**, 792 (1975) .

[10] H. Georgi, *Phys. Lett.* **240B**, 447 (1990) .

[11] M. Luke and A. V. Manohar, *Phys. Lett.* **286B**, 348 (1992) .

[12] A. Falk, B. Grinstein, and M. Luke, *Nucl. Phys.* **B357**, 185 (1991) .

[13] J. C. Collins, *Nucl. Phys.* **B149**, 90 (1979) .

[14] See *e.g.* E. Witten, *Nucl. Phys.* **B160**, 57 (1979) .

[15] N. Dorey, J. Hughes, and M. Mattis, hep-ph/9406406; A. V. Manohar, UCSD/PTH 94-14, hep-ph/9407211.

[16] S. J. Brodsky and G. Farrar, *Phys. Rev. Lett.* **31**, 1153 (1973) ; *Phys. Rev.* **D11**, 1309 (1975) ; S. J. Brodsky and G. P. Lepage, *Phys. Rev.* **D22**, 2157 (1980) . For a recent review, see S. J. Brodsky and G. P. Lepage in *Perturbative Quantum Chromodynamics*, edited by A. H. Mueller (World Scientific, 1989).

[17] J. F. Amundson *et. al.*, *Phys. Lett.* **296B**, 415 (1992) ; P. Cho and H. Georgi, *Phys. Lett.* **296B**, 408 (1992) .

[18] See *e.g.* B. Holdom, S. Jaimungal, R. Lewis, and M. Sutherland, UTPT-94-25, hep-ph/9410324; V. M. Belyaev, V. M. Braun, A. Khodjamirian, and R. Rückl, MPI-PHT/94-62, hep-ph/9410280.

[19] N. Isgur and M. B. Wise, *Phys. Lett.* **232B**, 113 (1989) ; *Phys. Lett.* **237B**, 527 (1990) .

[20] B. Grinstein and P. F. Mende, *Nucl. Phys.* **B425**, 451 (1994) .

[21] A. Duncan and A. H. Mueller, *Phys. Rev.* **D21**, 1636 (1980)

Figure Captions

Fig. 1. Contributions to the Compton amplitude T in the large-N limit. The sum over n runs over excited heavy mesons, while the sum over r and s runs over vector mesons. The shaded blobs on the graphs involving a sum over n are heavy meson form factors; see fig. 2.

Fig. 2. Contributions to heavy-meson form factors in the large-N limit. The sum over r runs over vector meson states.

Fig. 3. Contributions to the quark scattering amplitude used to determine the asymptotic behavior of the light-quark current form factors. The thick line is the heavy quark, and the curly line represents the gluon propagator.

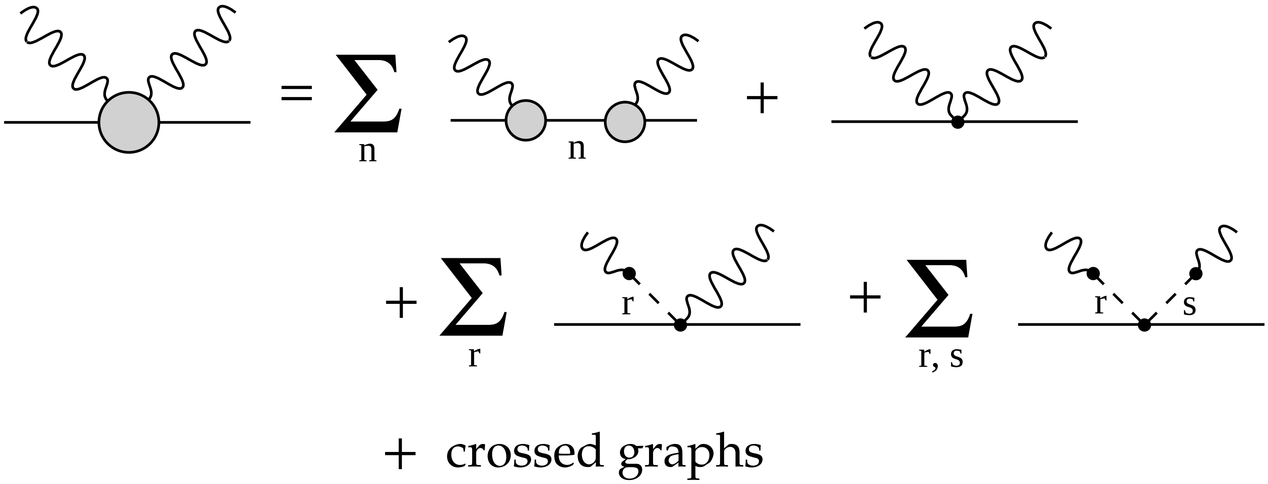

fig. 1

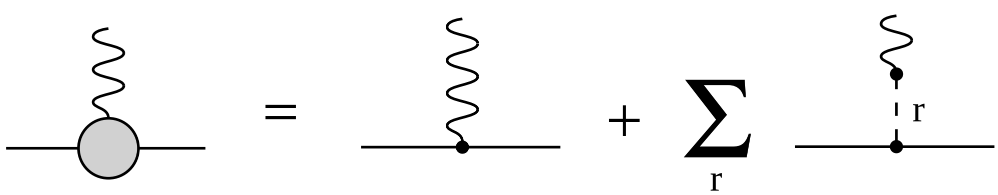

fig. 2

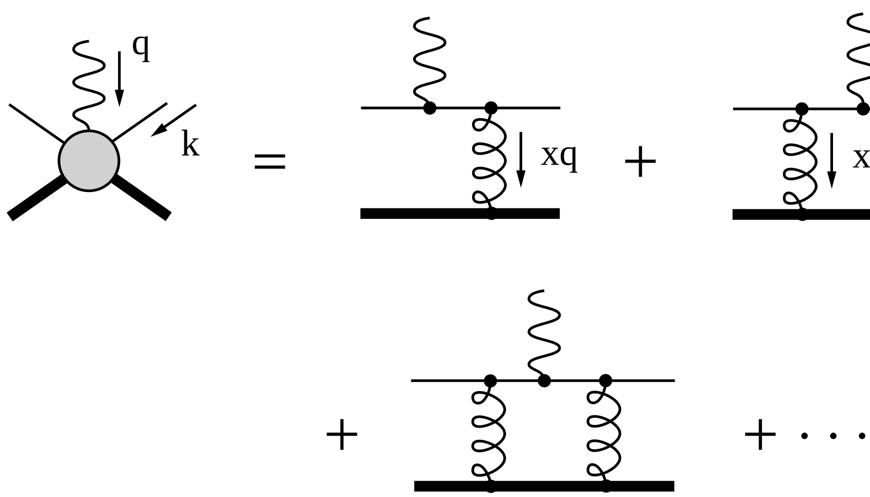

fig. 3